Track Planarity Testing and Embedding

Christian Bachmaier
bachmaier@fmi.uni-passau.de

Franz J. Brandenburg
brandenb@fmi.uni-passau.de

Michael Forster
forster@fmi.uni-passau.de

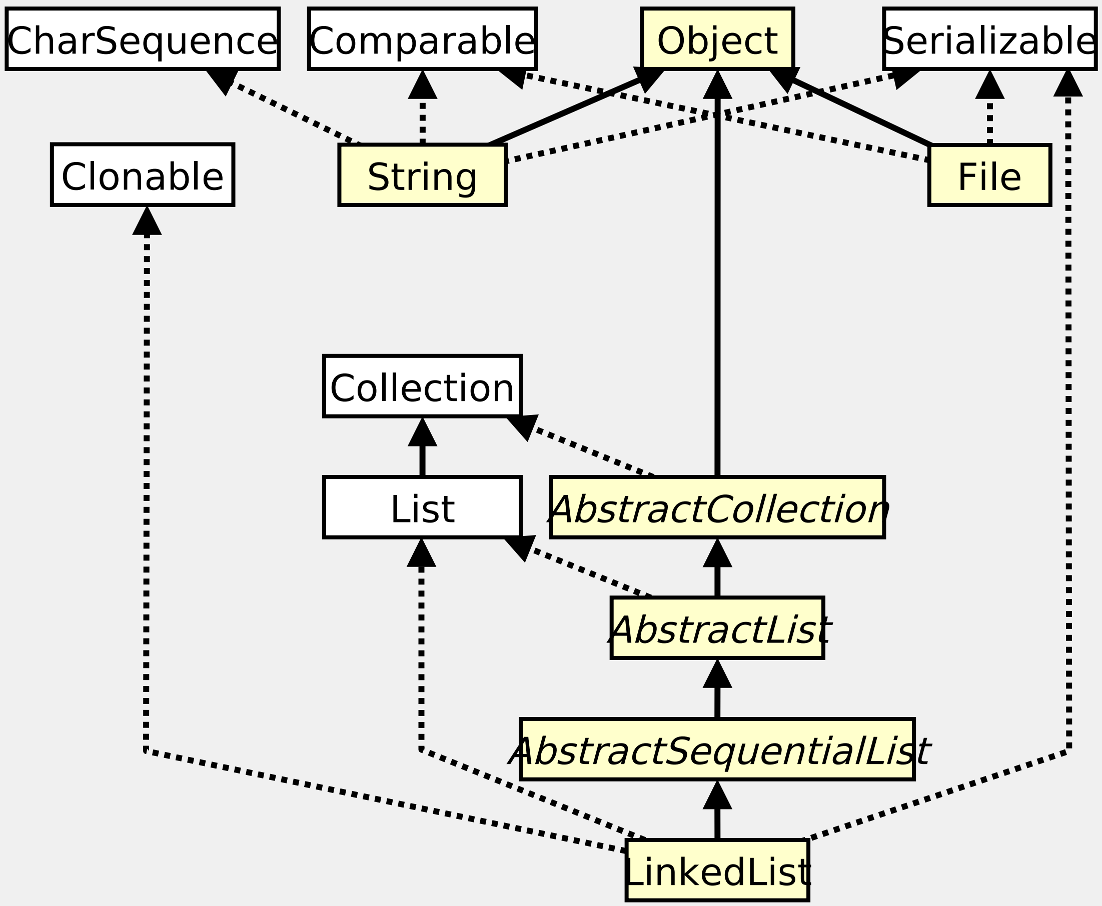
University of Passau, Germany

Motivation

- Graphs are abstract models of relations in the real world
- Drawing graphs planar
- Planarity test in O(n) of [Lempel, Even, Cederbaum (LEC) 67]
- Directed acyclic graphs
  - Scheduling
  - Flow charts
  - ER diagrams
  - UML diagrams

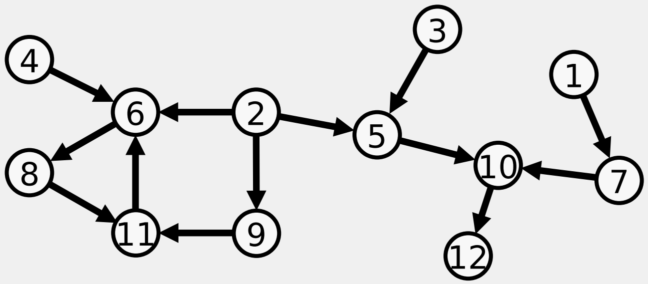
UML Diagrams

- Vertices
  - Classes
  - Interfaces
- Directed Edges
  - Inheritance
  - Association



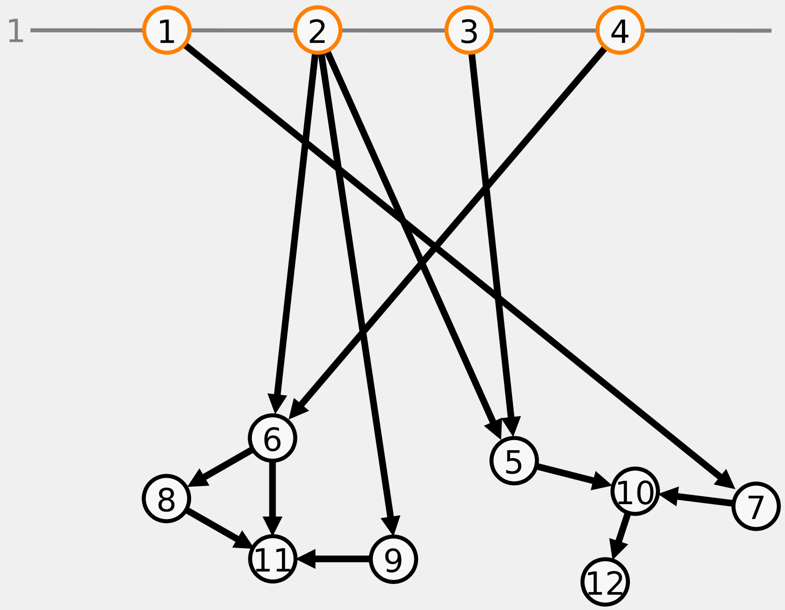
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling



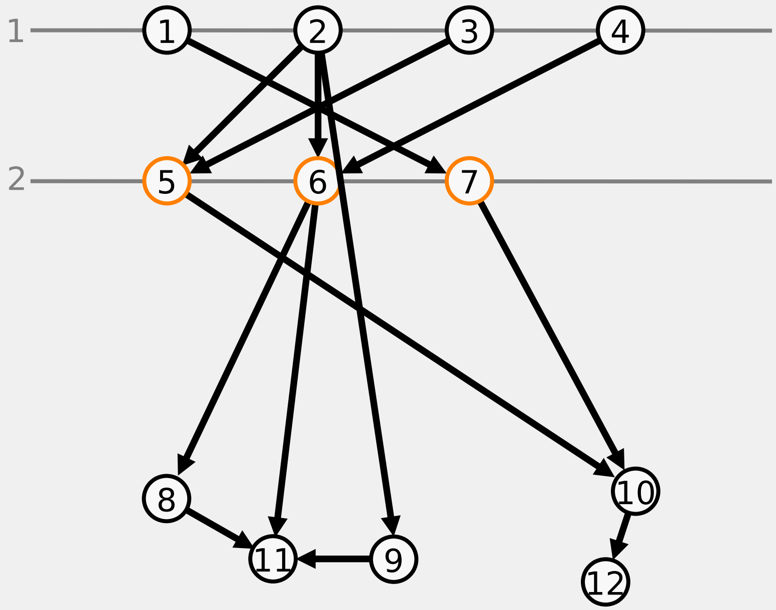
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling



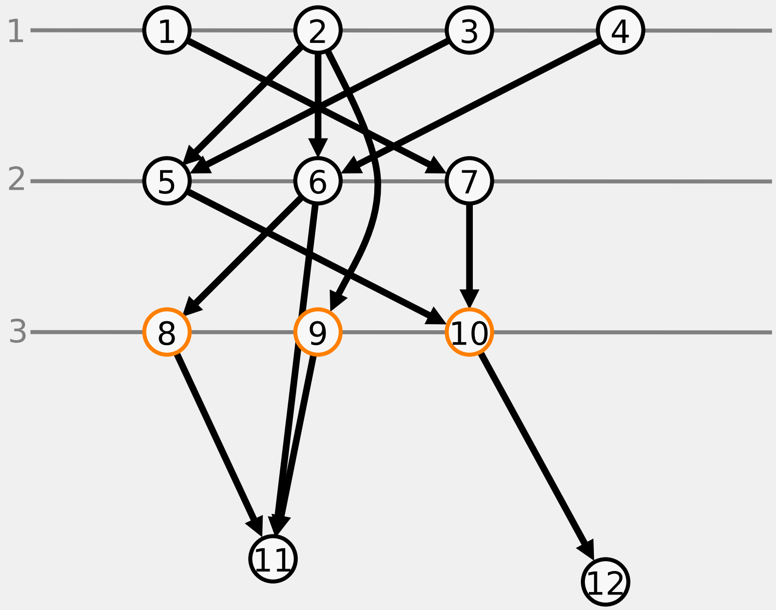
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling



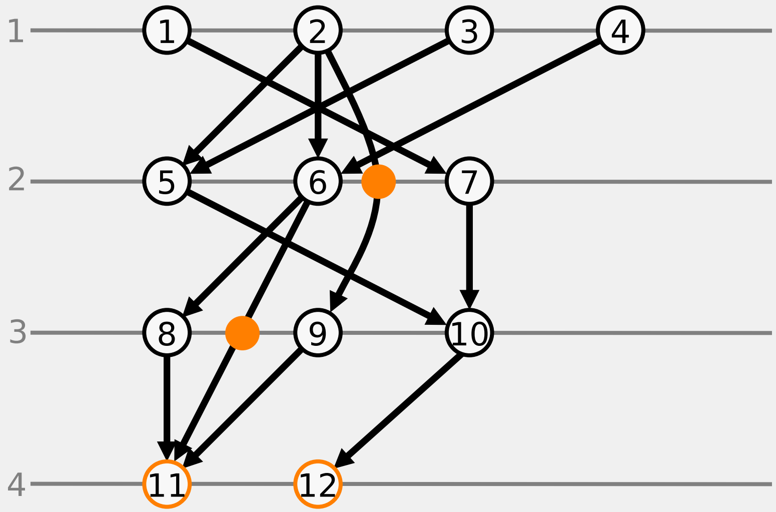
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling



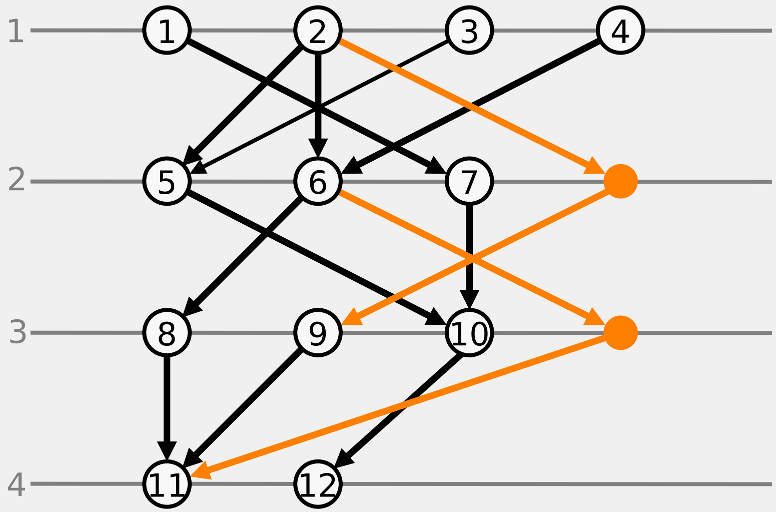
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges



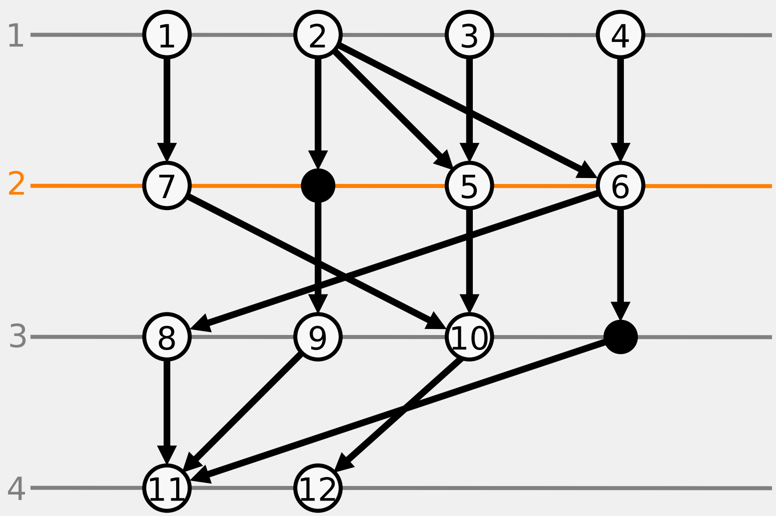
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings



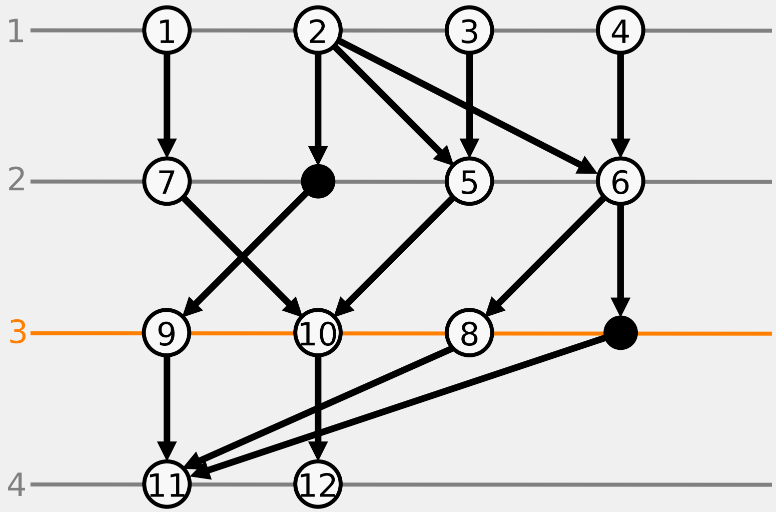
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings



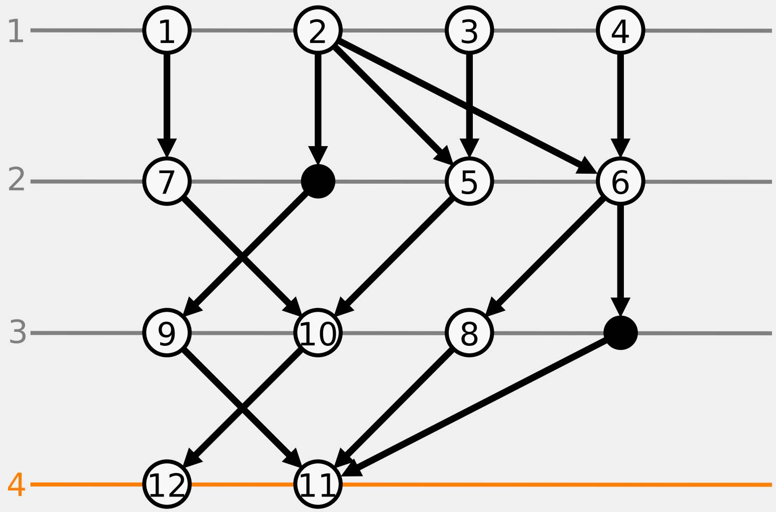
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings



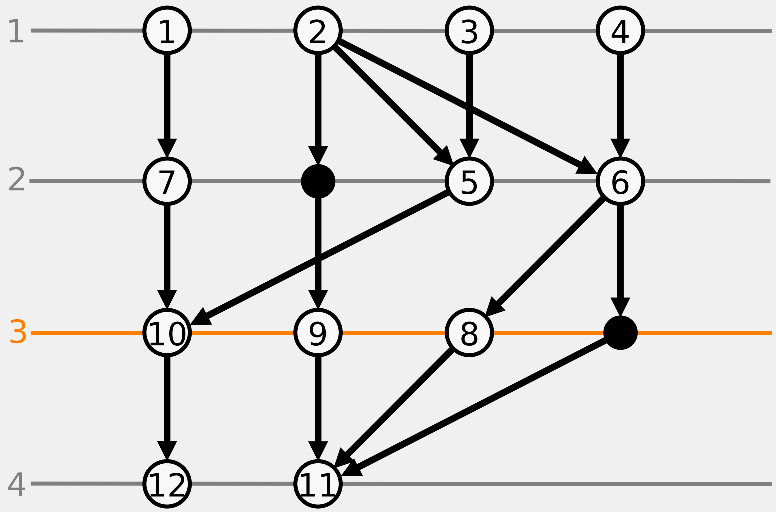
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings



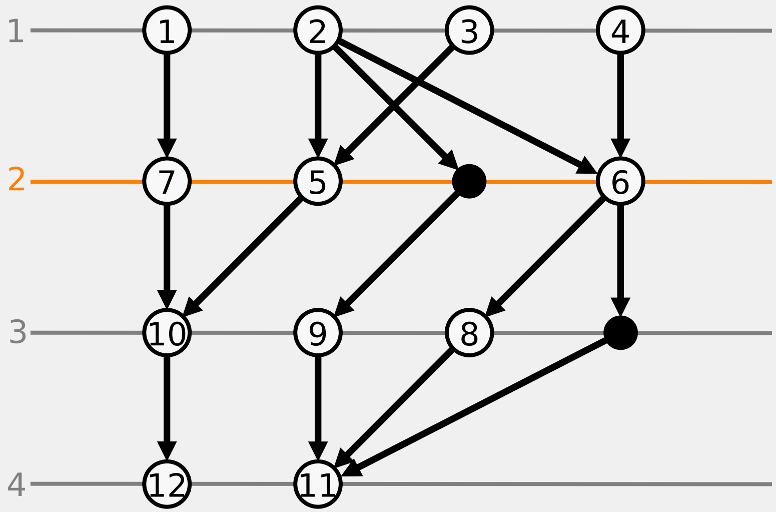
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings



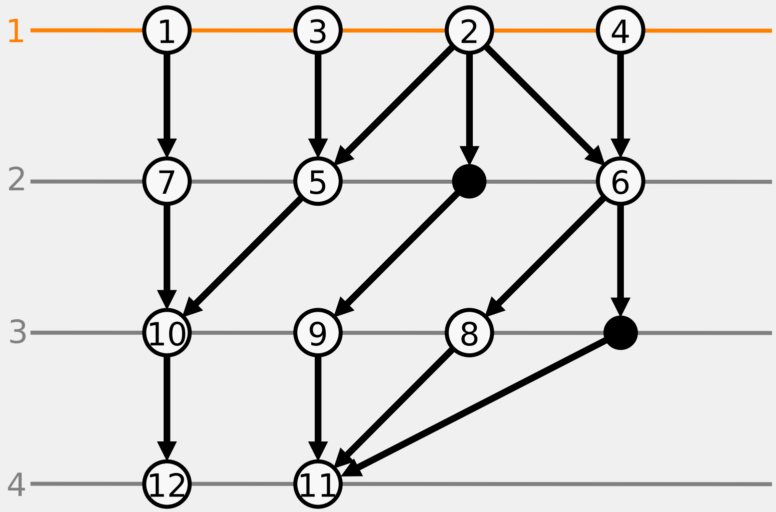
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings



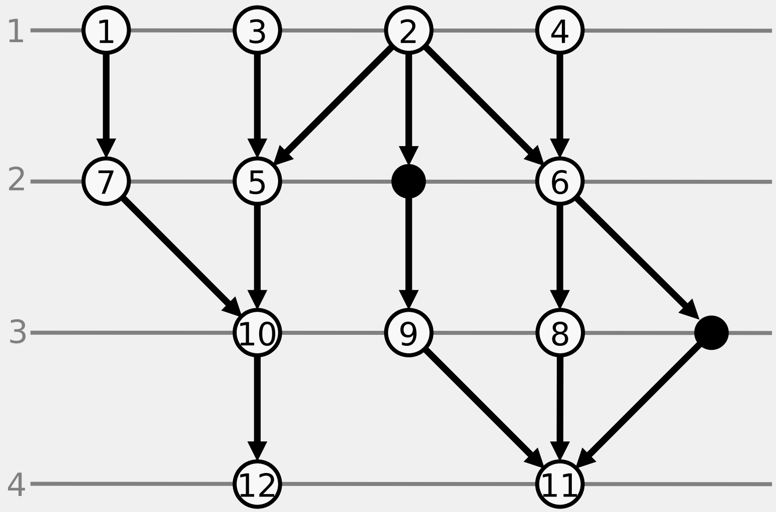
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings
  - Assign x-coordinates



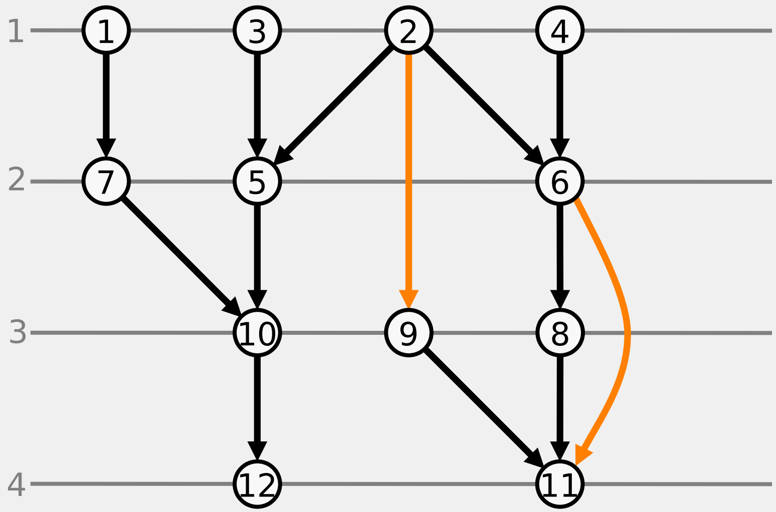
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings
  - Assign x-coordinates
  - Join long edges



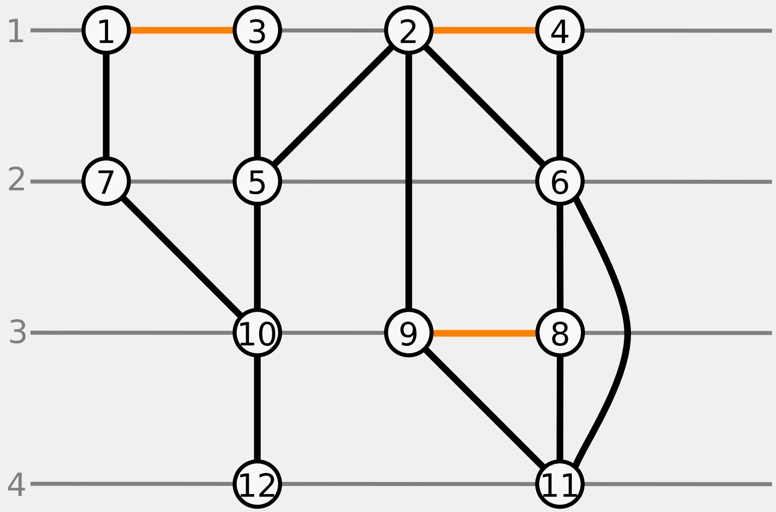
Sugiyama Algorithm

- Algorithm
  - Remove cycles
  - Generate levelling
  - Split long edges
  - Reduce crossings
  - Assign x-coordinates
  - Join long edges
- Basic properties
  - 2 level crossing minimisation
  - Minimum edge set whose removal eliminates crossings
  - NP-hard
- Is there a drawing without crossings?



Idea

- Level planarity solved
  - Needs O(n) time
  - Levelling given
  - Implicit edge directions



- Planarity with horizontal edges

Overview

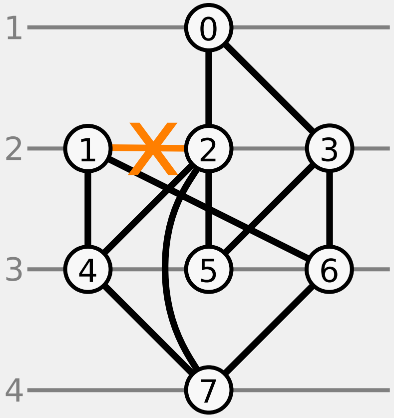
- Motivation
- Level graphs
  - Level planar graphs
  - Related Work
    - Level planarity testing & embedding
- Track graphs
  - Track planar graphs
- Reduction of track planarity to level planarity
- Track planarity testing & embedding
- Circle planarity testing & embedding

Level Planarity

Definition, Example, Previous Work

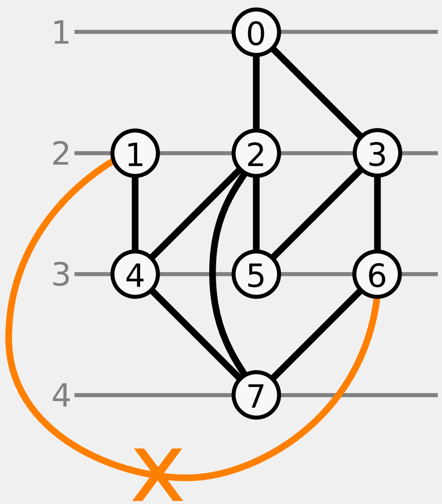
Level Graph

- A graph $G = (V_1 \cup V_2 \cup \ldots \cup V_k, E)$ is a k-level graph
  - Vertex partitioning into k disjoint levels
  - No horizontal edges
- G is proper
  - Each edge between adjacent levels



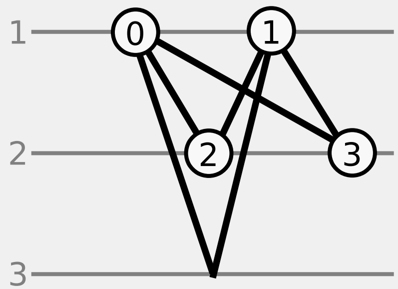
Level Planar Graph

- A k-level graph is k-level planar
  - Edges drawn strictly downwards
  - Planar
- Level planar embedding
  - Vertex ordering for each level $\leq_i$



Levelling

- Level planarity depends on levelling
- Given Levelling
  - NP-hard for proper graphs [Heath, Rosenberg 92]



Level Planarity Testing & Embedding

- [Jünger, Leipert, Mutzel (JLM) 98 and 99] O(n) time
  - Fastest algorithm
  - Based on PQ-tree data structure
  - Similar to vertex addition method for testing planarity (LEC)
- [Healy and Kuusik 99] $O(n^2)$ time
  - Simpler than the above
  - Only for proper graphs
  - Embedding in $O(n^3)$ time

Extensions

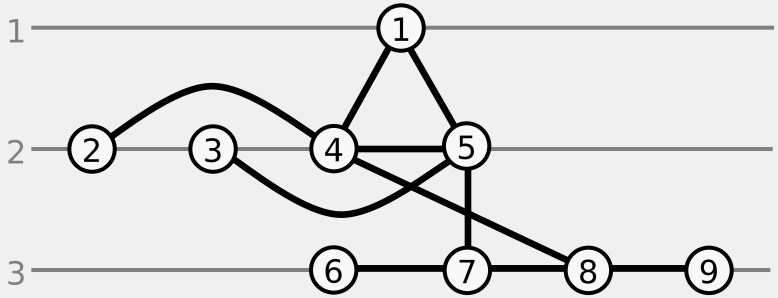
- Planar
- Level Planar
- Extensions
  - Track planar
  - Radial level planar
  - Circle planar
  - Clustered level planar (next talk)

Track Planarity

Definitions, Example

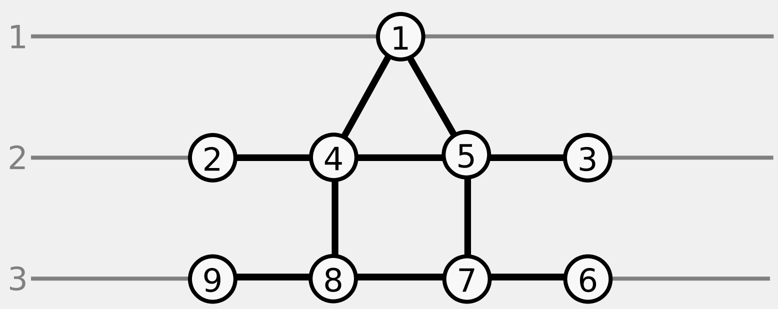
Track Graph

- A graph $G = (V_1 \cup V_2 \cup \ldots \cup V_k, E \cup E')$ is a k-track graph
  - $(V_1 \cup V_2 \cup \ldots \cup V_k, E)$ is a k-level graph
  - Horizontal edges E' within the same level (track)



Track Planar Graph

- A k-track graph is k-track planar
  - Without horizontal edges k-level planar
  - Horizontal edges connect consecutive vertices according to a $\leq_i$
    - No vertex between two end vertices of a horizontal edge
  - All edges are drawn weak monotonic downwards

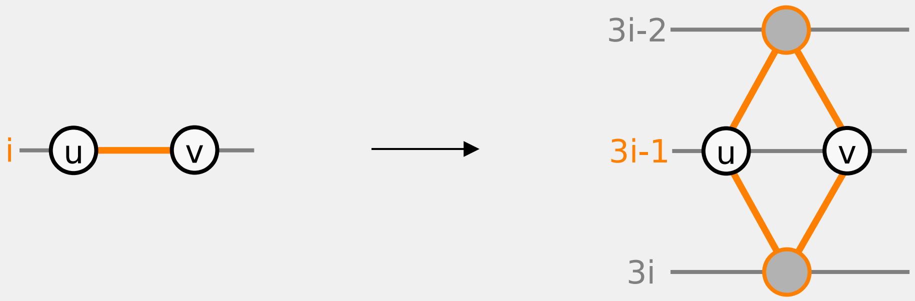


Reduction of Track Planarity to Level Planarity

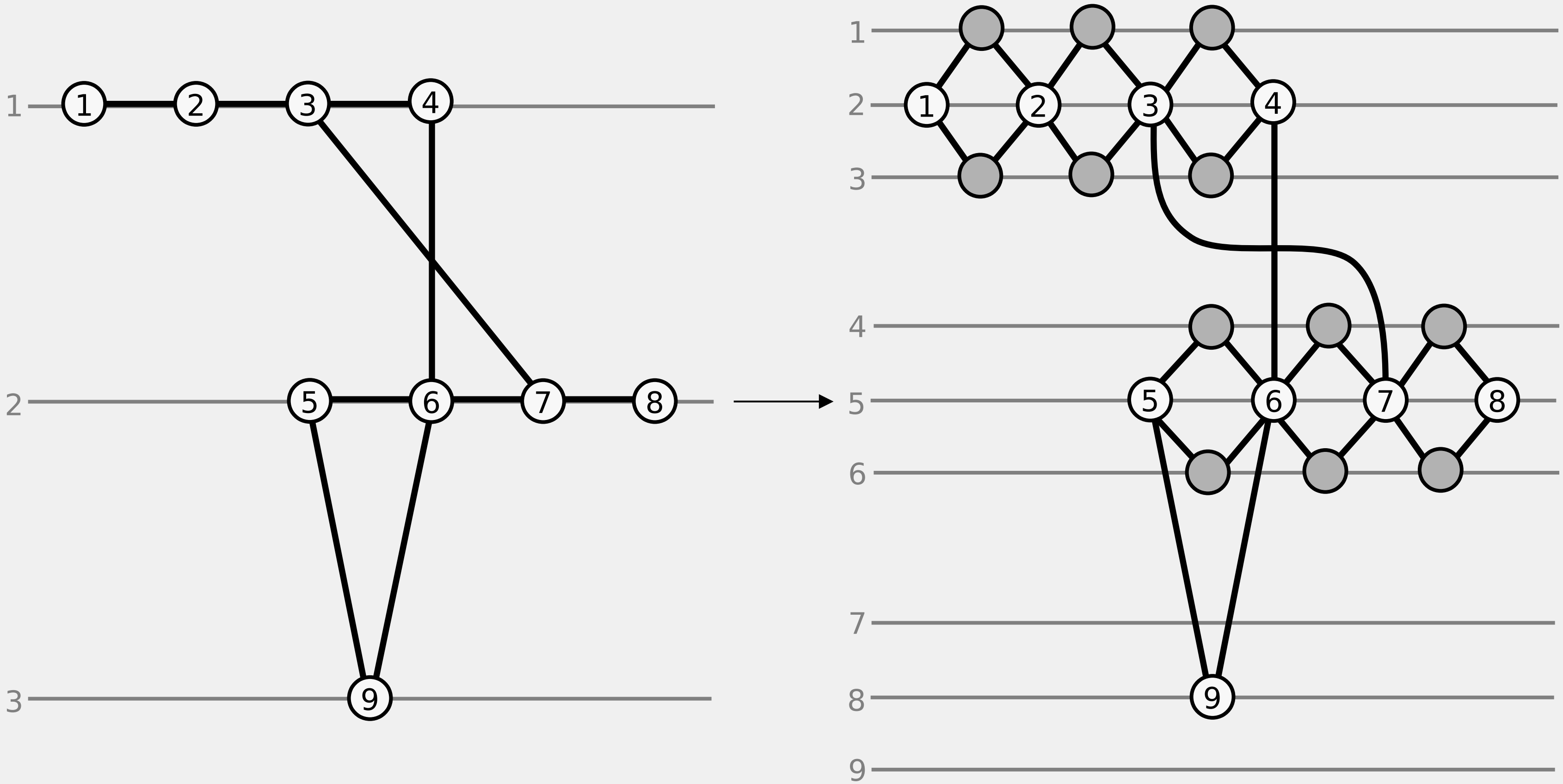
Idea, Example

Transformation

- Triple levels
- Replace each horizontal edge by a diamond

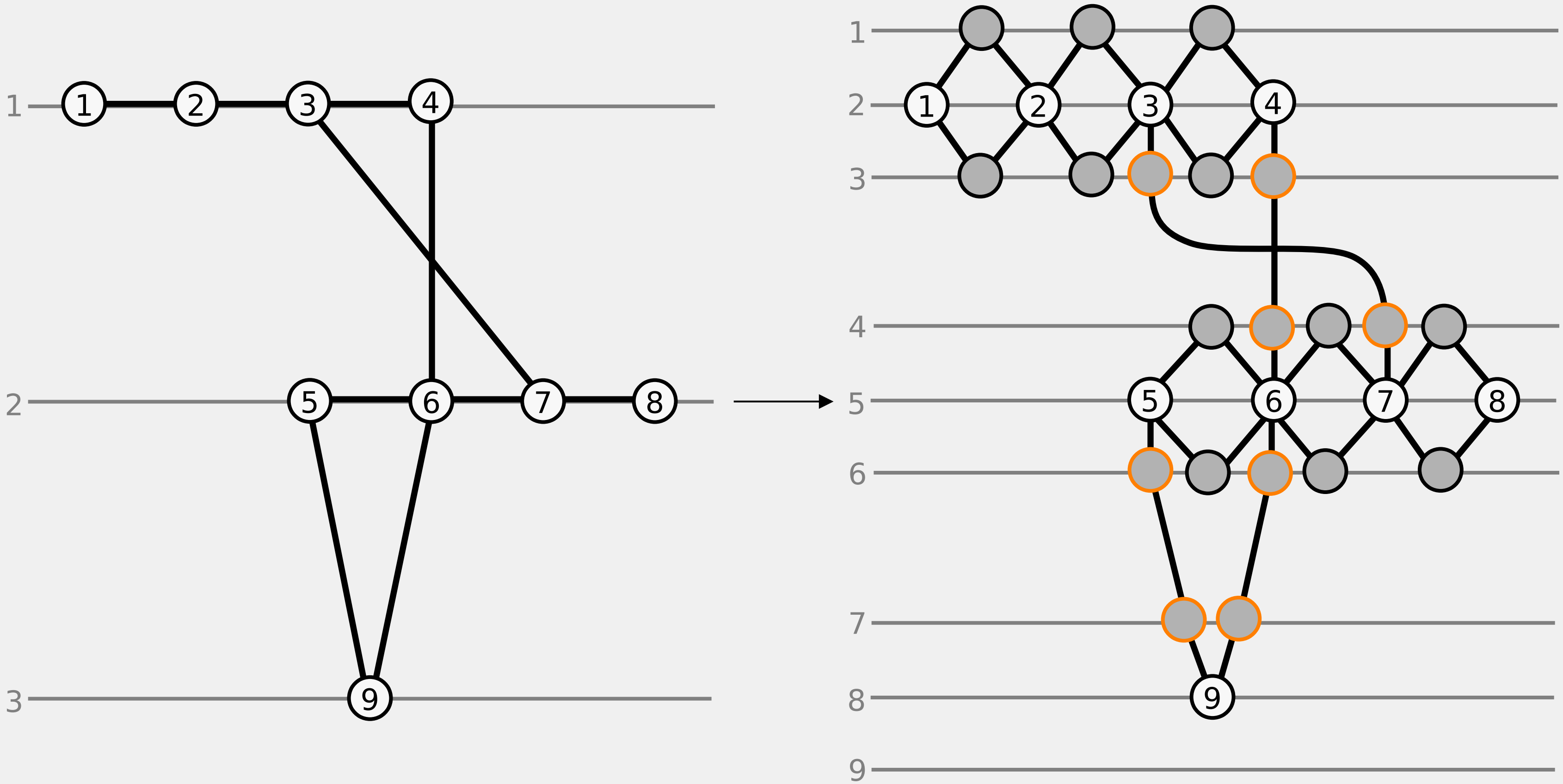


- Lemma
  - Transformation can be done in O(n) time
  - Transformed graph has O(n) size
- W.l.o.g. G contains no isolated vertices



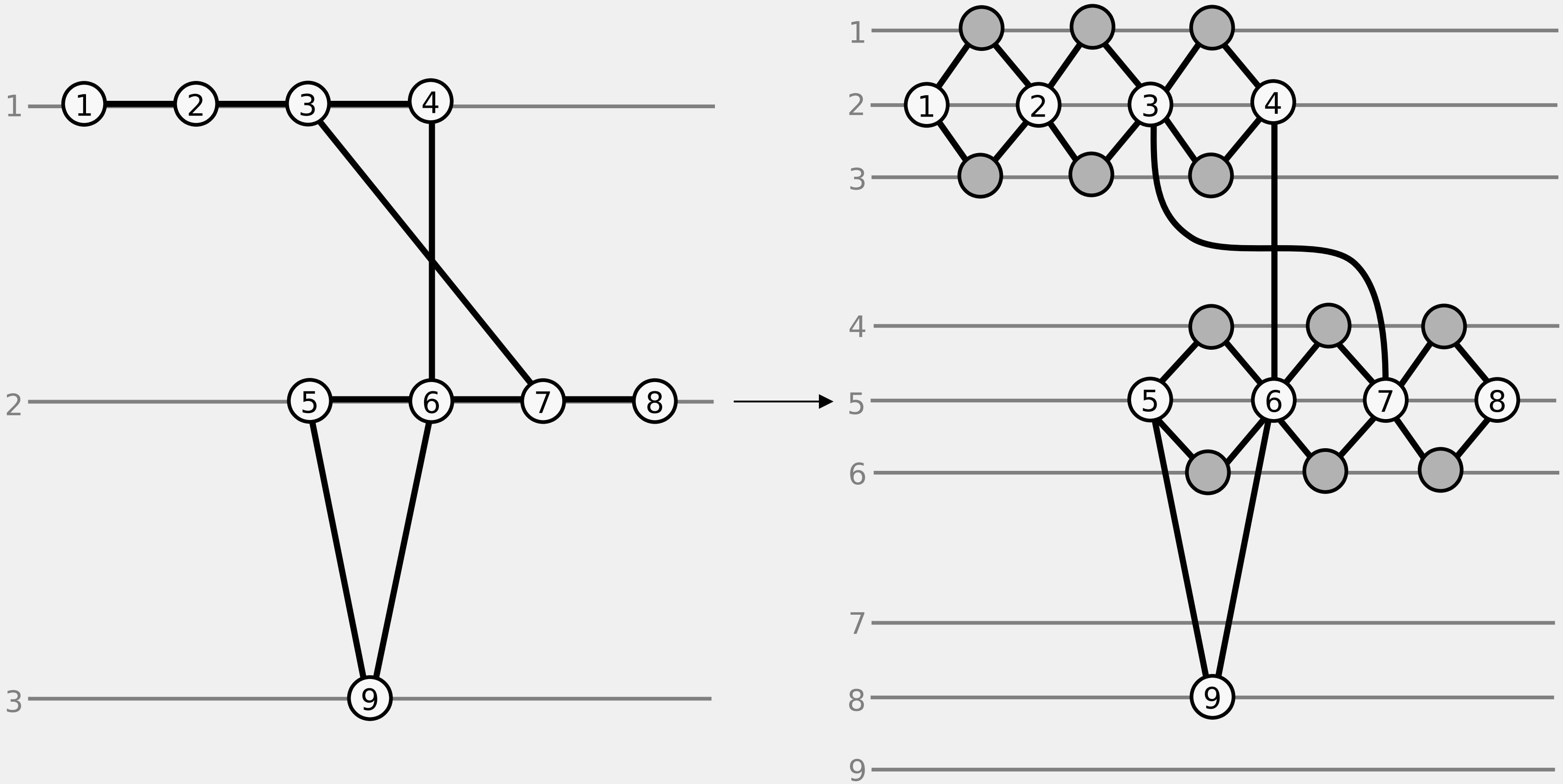
- Generates long edges even for proper graphs

Transformation Example



- Generates long edges even for proper graphs

Transformation Example

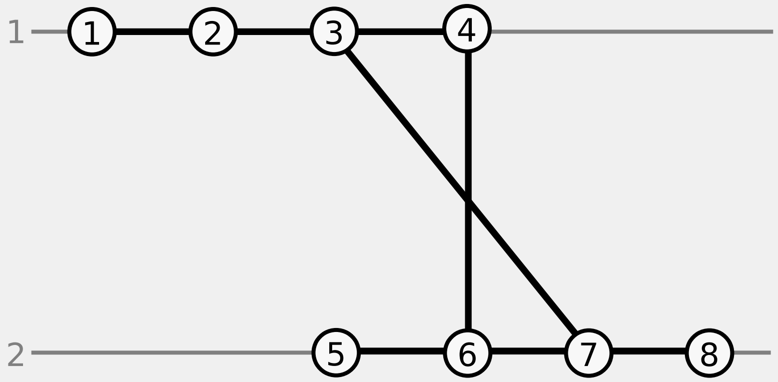


- Generates long edges even for proper graphs
- Planarity and embedding is preserved

Reduction

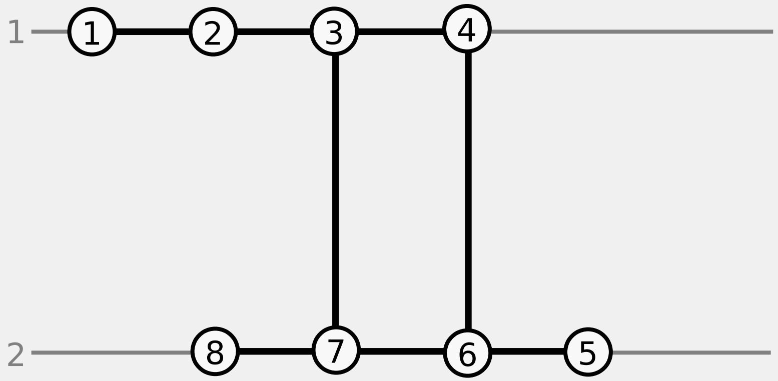
- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar

Proof
  - "⇒"
    - G is k-track planar



Reduction

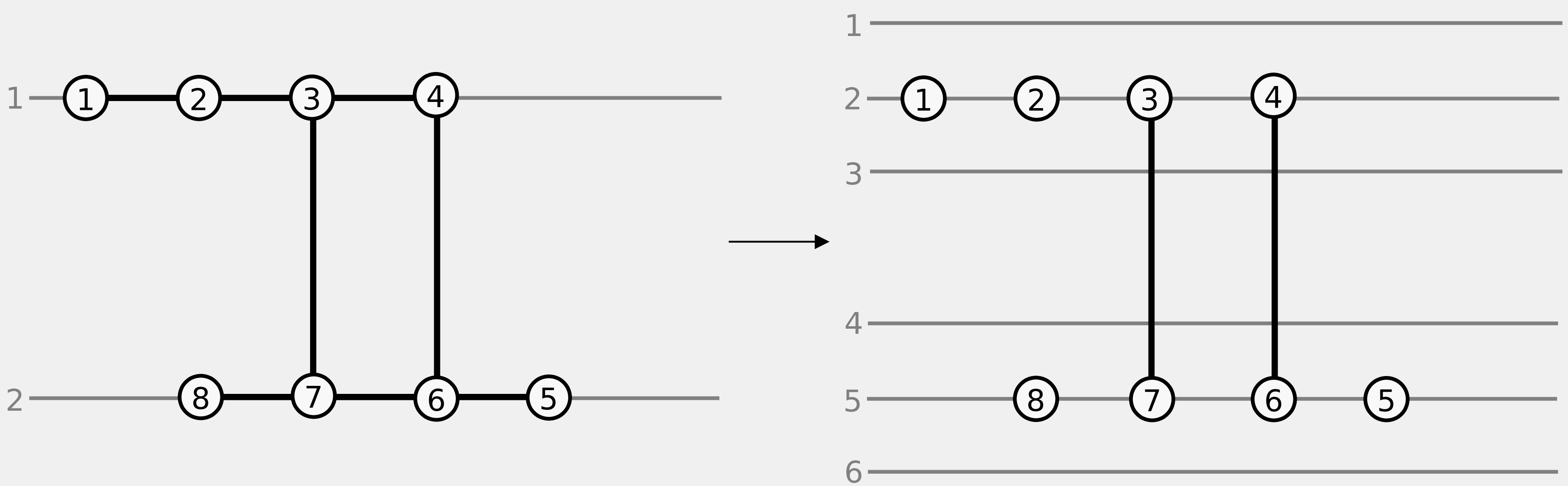
- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar
- Proof
  - "⇒"
    - G has a k-track planar embedding



Reduction

- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar

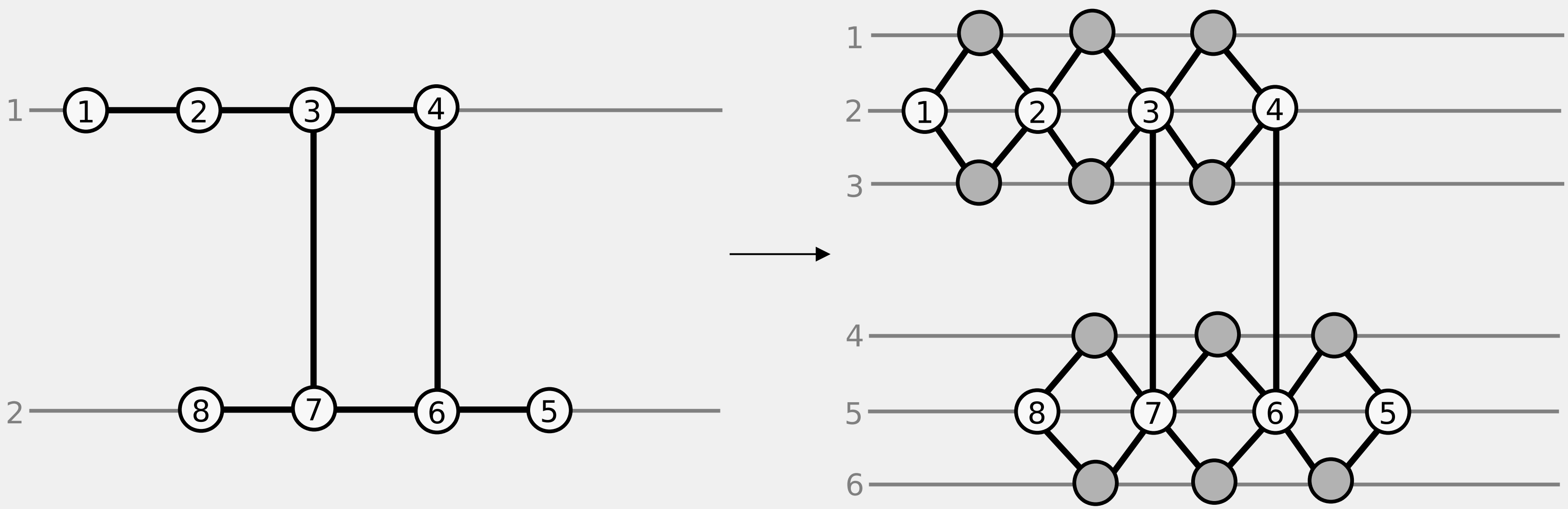
Proof
  - "⇒"
    - Construct a level planar embedding with $\leq'_{3i-1} = \leq_i$



Reduction

- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar

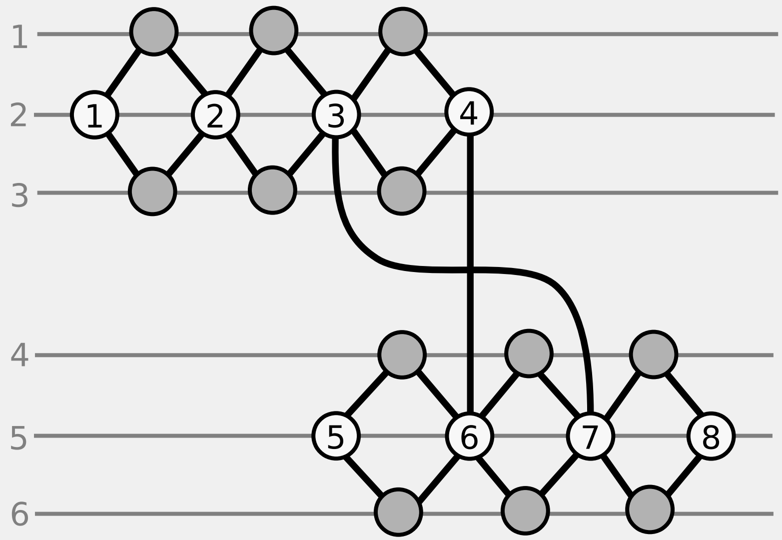
  Proof
  - "⇒"
    - Orderings of dummy vertices defined by adjacent original vertices



Reduction

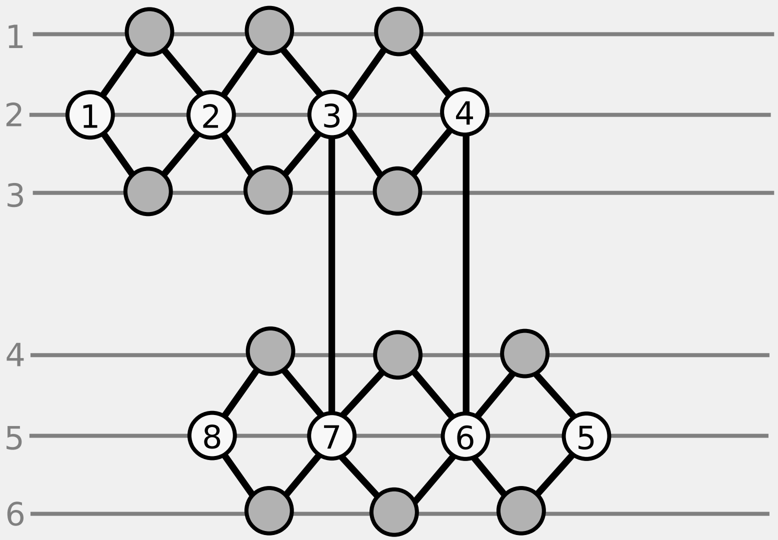
- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar

Proof
  - " ⇐ "
    - G' is 3k-level planar



Reduction

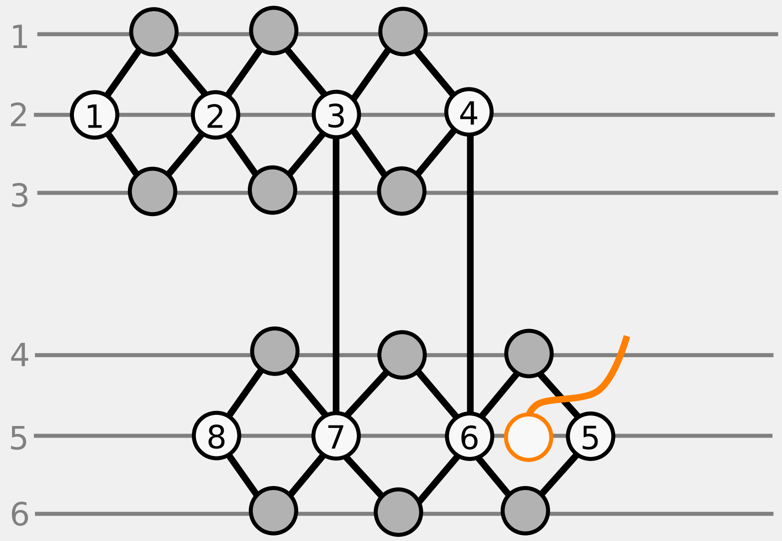
- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar
- Proof
  - " ⇐ "
    - G' has a 3k-level planar embedding



Reduction

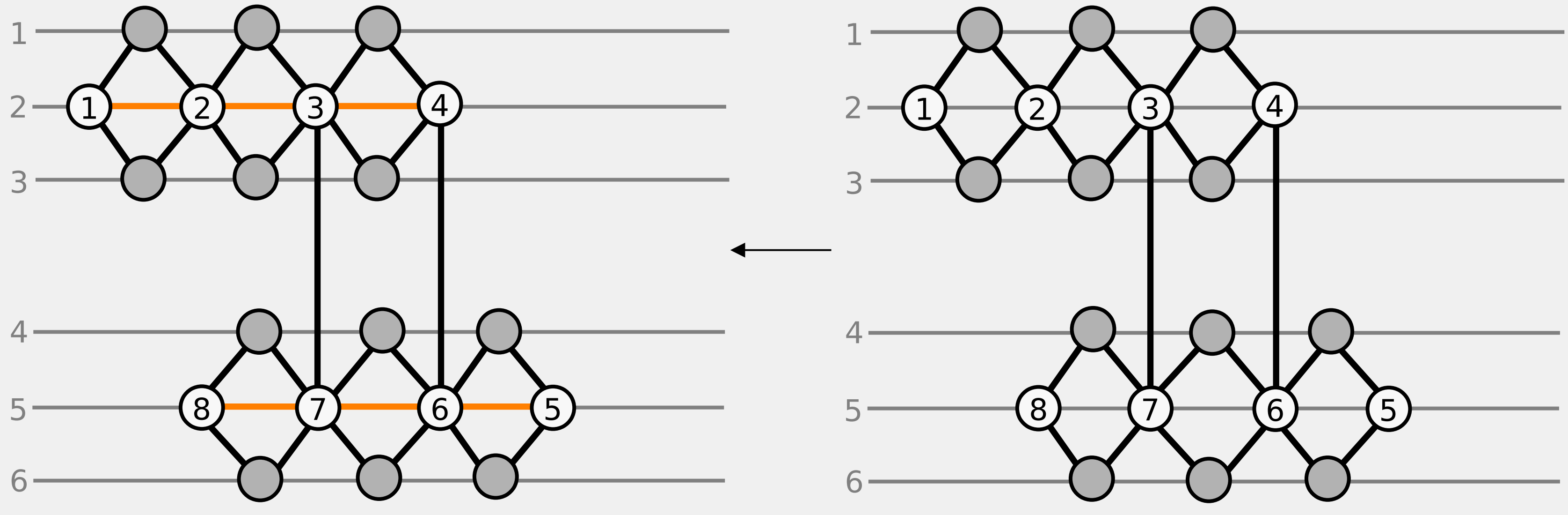
- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar

Proof
  - " ⇐ "
    - The inner face of every diamond is empty



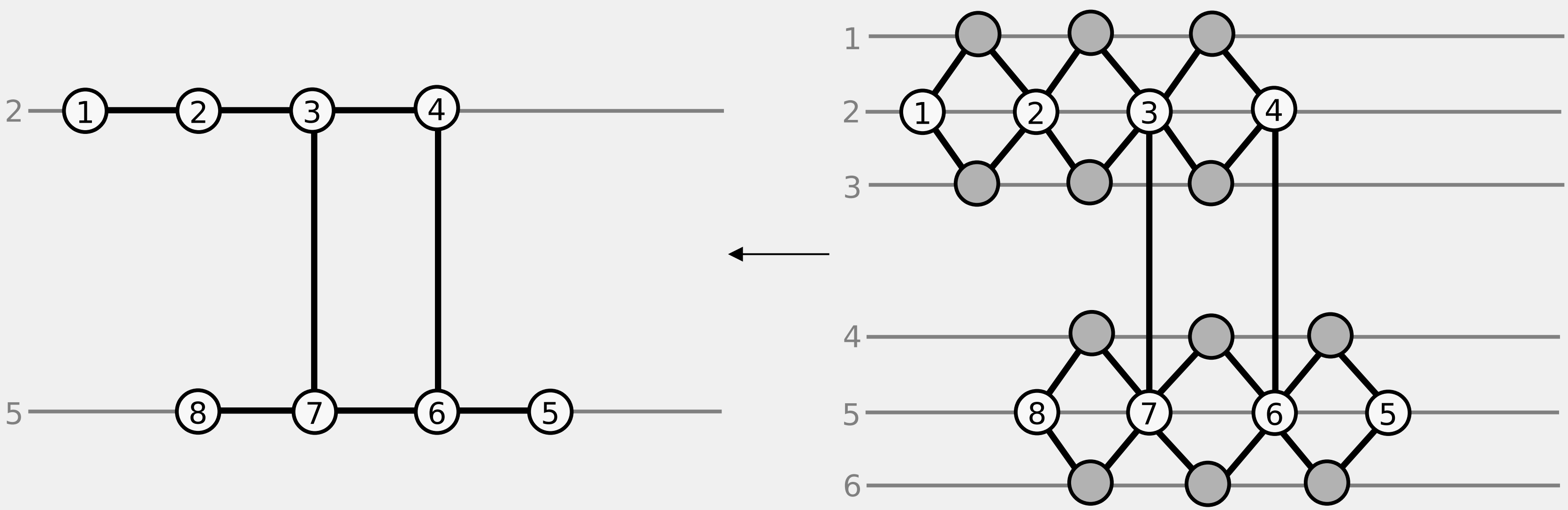
Reduction

- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar
- Proof
  - " ⇐ "
    - Create an edge between the original vertices of every diamond



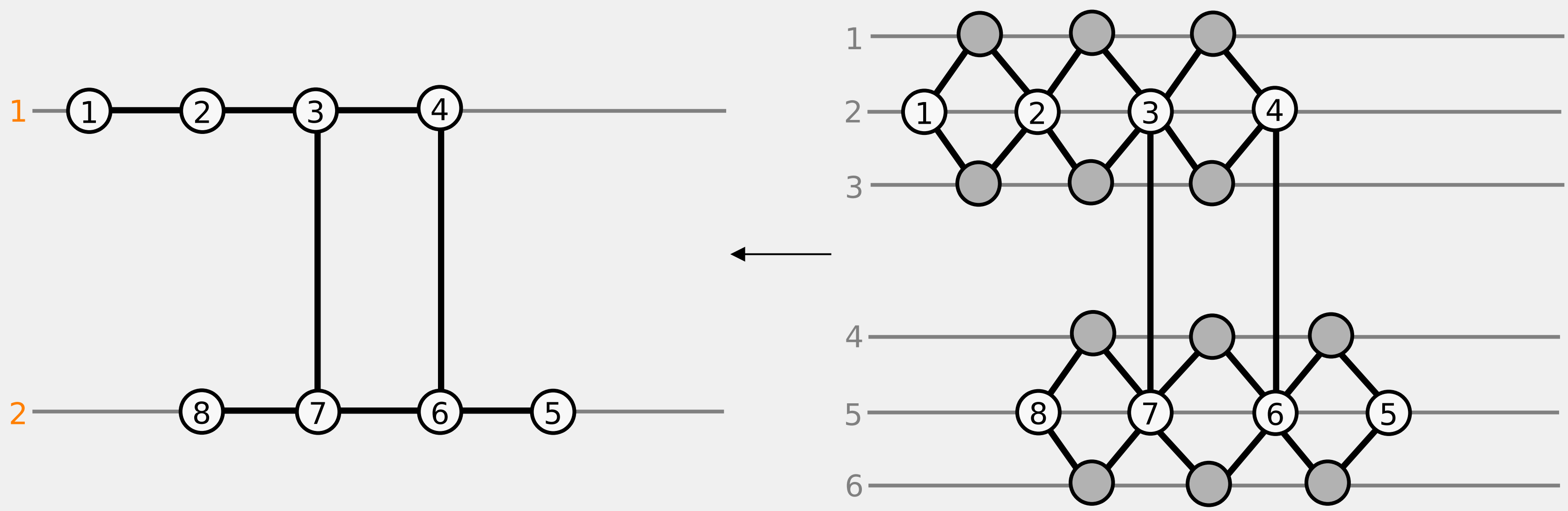
Reduction

- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar
- Proof
  - " ⇐ "
    - Discard dummy levels



Reduction

- Lemma
  - Let G be a k-track graph and G' its 3k-level transformation
    - G is k-track planar ⇔ G' is 3k-level planar
- Proof
  - " ⇐ "
    - Renumber remaining levels



Reduction...

- Theorem
  - There is an O(n) time reduction of track planarity to level planarity

Algorithm for Track Planarity Testing and Embedding

Automated Detection

Algorithm

- Transform the k-track graph in a 3k-level graph
  - O(n) for non-proper case
- Any level planarity testing algorithm can be used
  - Construction must be made proper for [Healy and Kuusik 99]
  - Dominates time complexity of track planarity testing

Complexity

- Corollary
  - There is an O(n) time algorithm to decide k-track planarity of a graph
- Corollary
  - There is an O(n) time algorithm for computing a k-track planar embedding for a k-track planar graph
    - The algorithm of JLM computes a level embedding in O(n) time
    - Treat this embedding like shown in the proof of the last lemma

Circle Planarity

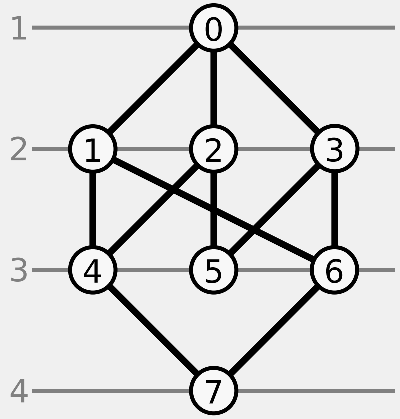
Definition, Idea, Example, Result

Radial Level Planar Graphs

- Generalisation of level planar graphs
- Level graph G is radial k-level planar if it can be drawn such that
  - Vertices of each level lie on a concentric circle
  - Edges drawn strictly outwards
  - Planar

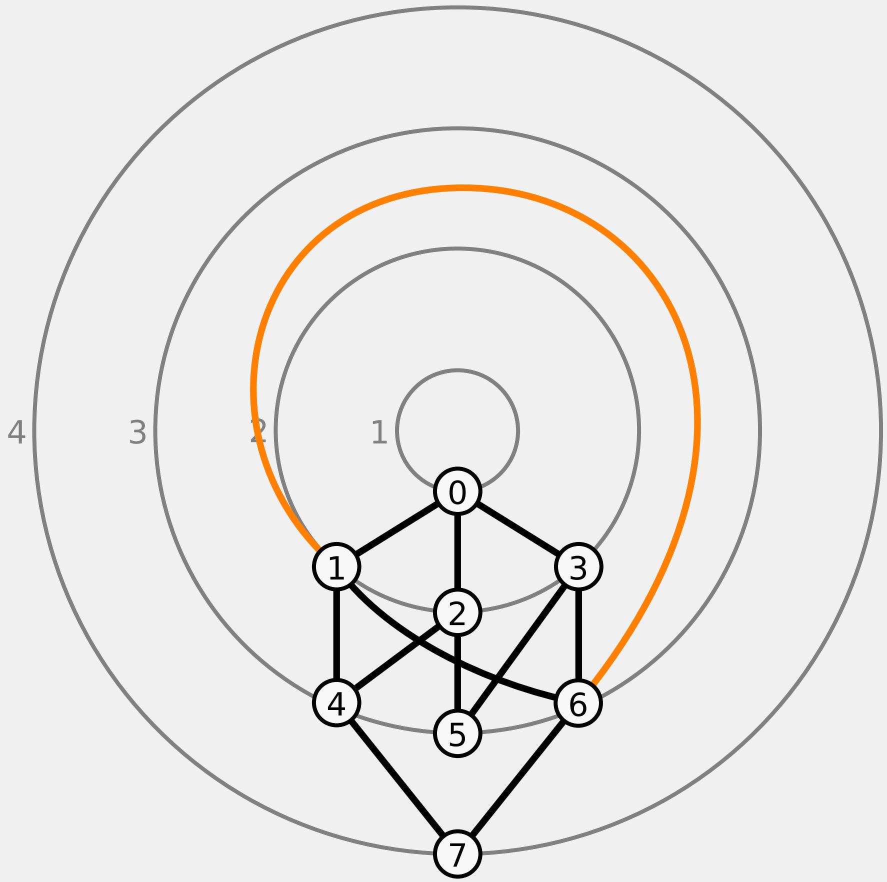
Transformation

- 4-level graph
- Not level planar
- Radial 4-level planar



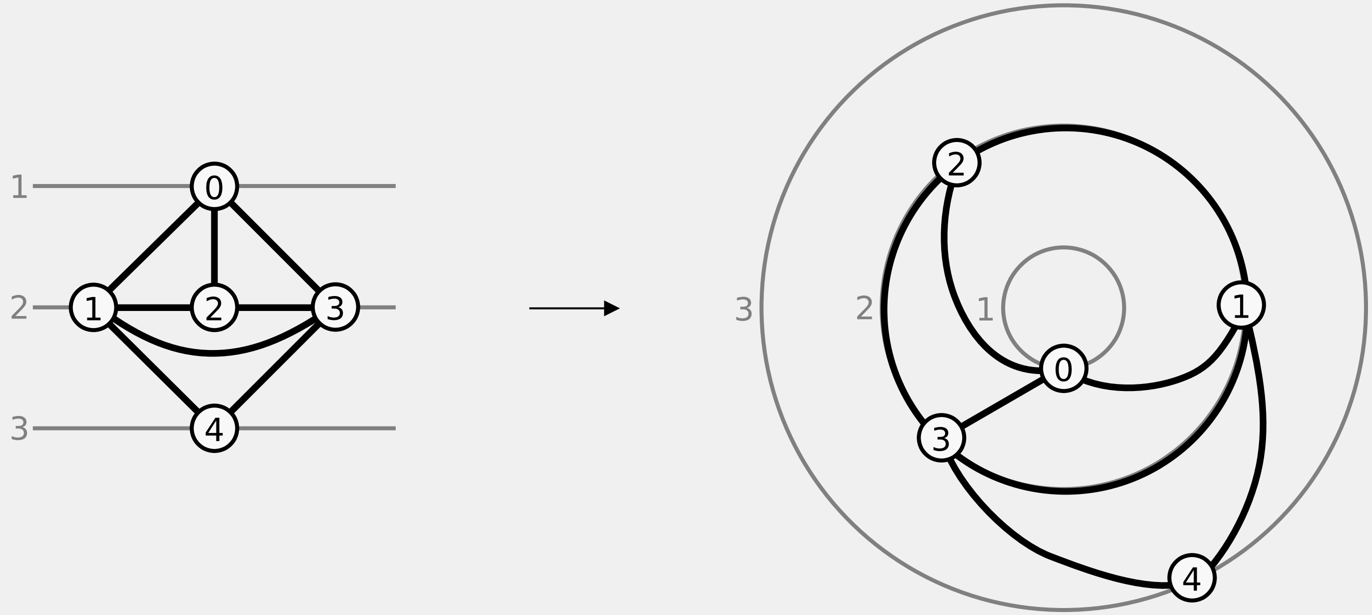
Transformation

- 4-level graph
- Not level planar
- Radial 4-level planar
- Bend level lines to circles
- Planar possibility to route edge (1, 6)
- Testing and embedding in O(n)
  - [Bachmaier, Brandenburg, Forster 2003]



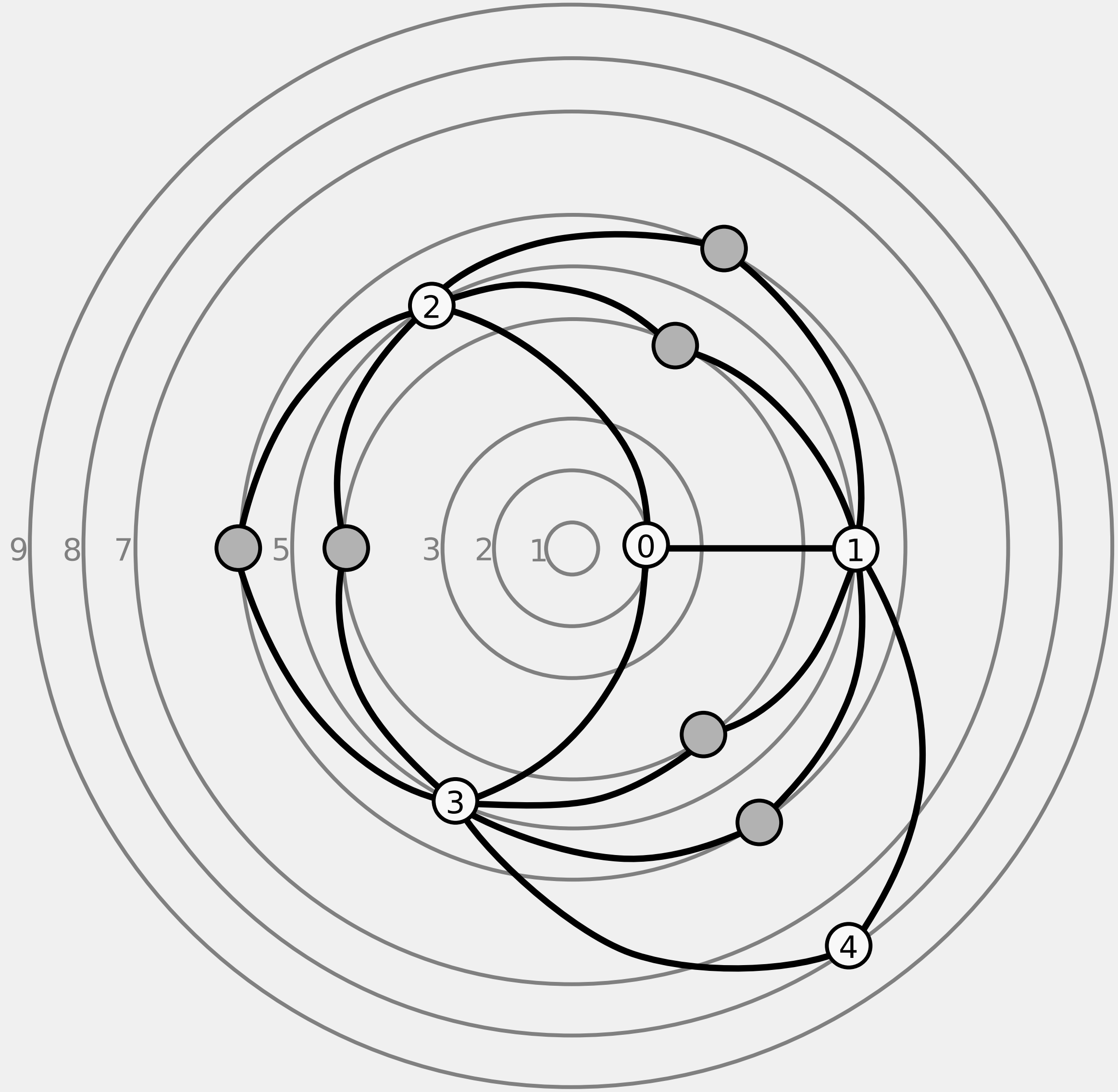
Circle Planar Graph

- Combination of radial planarity and track planarity
  - Radial planar graphs with inner level edges



- Idea
  - Transformation into radial 3k-level graph
- Isolated vertices
  - Postprocessing step

Circle Planarity



Complexity

- Lemma
  - Let G be a k-circle graph and G' its radial 3k-level transformation
    - G is k-circle planar ⇔ G' is radial 3k-level planar
- Theorem
  - There is an O(n) time reduction of circle planarity to radial level planarity
- Corollary
  - There is an O(n) time algorithm for
    - Deciding k-circle planarity of a graph
    - Computing a k-circle planar embedding

Remarks

Past and Future Work

Summary

- More graphs having its vertices assigned to levels can be drawn nicely
- Prototypic implementation
  - C++
  - Using GTL
  - Following the technique of JLM
  - Feasibility study
  - Proof of concept
  - Understanding all technical details

Future Work

- Assigning Coordinates
  - Drawing
  - Few edge bends
- Detection of Minimum (Level|Track) Non-Planar subgraph patterns
  - Level variants of Kuratowski graphs $K_{3,3}$ and K_5
  - MLNP pattern
  - MTNP pattern
  - Patterns for radial cases

Thank you!